

# Leptonic contributions to the effective electromagnetic coupling at four-loop order in QED

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## Abstract

The running of the effective electromagnetic coupling is for many electroweak observables the dominant correction. It plays an important role for deriving constraints on the Standard Model in the context of electroweak precision measurements. We compute the four-loop QED corrections to the running of the effective electromagnetic coupling and perform a numerical evaluation of the different gauge invariant subsets.

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# 1 Introduction

The input parameters of Quantum electrodynamics (QED), like the fine-structure constant or the masses of the charged leptons are known with high precision. For example, the fine-structure constant has been measured at the level of  $\sim 0.32$  ppb [1] or the mass of the electron in units of MeV is known at the level of  $\sim 22$  ppb [1]. Also from theory side physical quantities in QED can be computed to high accuracy in perturbation theory, since the electromagnetic coupling constant is a small parameter.

QED corrections play an important role for electroweak precision observables where the effective electromagnetic coupling at the  $Z$ -boson mass scale enters. The change from the fine-structure constant  $\alpha$ , defined at zero momentum transfer, to the effective electromagnetic coupling at the  $Z$ -boson mass scale is mediated by the photon vacuum polarization function. At the  $Z$ -boson mass scale the effective electromagnetic coupling is governed by the parameter  $\Delta\alpha$ , which plays an important role in constraining the Standard Model (SM) in the context of electroweak precision measurements, so that its precise theoretical understanding is an important task.

The parameter  $\Delta\alpha$  receives leptonic and hadronic contributions. The latter escape a perturbative treatment and are determined using cross-section measurements of electron positron annihilation into hadrons, see e.g. Refs. [2–9]. The hadronic contributions  $\Delta\alpha_{\text{had}}$  constitute currently the dominant uncertainty to  $\Delta\alpha$ . The leptonic contributions  $\Delta\alpha_{\text{lep}}$  at one- and two-loop order in QED are known from the results for the vacuum polarization function of Refs. [10, 11]. The three-loop order has been computed in Ref. [12] in terms of expansions in the small mass ratio  $M_\ell^2/M_Z^2$ , where  $M_\ell$  stands for the masses of the SM charged leptons and  $M_Z$  is the  $Z$ -boson mass. The dominant contribution to  $\Delta\alpha_{\text{lep}}$  arises from large logarithms in this small mass ratio, so that it is often sufficient to perform the expansion up to the constant term, including it, and to neglect power suppressed terms of higher order in the mass ratio  $M_\ell^2/M_Z^2$ . Within this work we will extend the calculation of  $\Delta\alpha_{\text{lep}}$  to four-loop order in QED, also in form of small lepton mass expansions up to the constant term. For the lower orders in perturbation theory we include the power suppressed terms. Next to the effective electromagnetic coupling we will also present an numerical evaluation of  $\Delta\alpha_{\text{lep}}$ , which allows to study the size of the different contributions. As a by-product of our calculation we will also provide the result for the relation between the  $\overline{\text{MS}}$  and on-shell mass from the contribution due to heavy fermion loops at three-loop order in QED.

In the next Section 2 we start with some generalities and the definition of our notation. In Section 3 we give an outline of our calculation. The new results for the vacuum polarization function in the on-shell scheme at four-loop order are presented in Section 4 including a numerical evaluation of  $\Delta\alpha_{\text{lep}}$ . Finally we close with our summary and conclusions in Section 5. In the Appendix we provide known results as supplementary information.

## 2 Generalities and notation

We consider the photon vacuum polarization in QED with all one particle irreducible insertions into the photon propagator. The vacuum polarization tensor is given by

$$\Pi^{\mu\nu}(q, M) = i \int dx e^{iqx} \langle 0 | T j^\mu(x) j^\nu(0) | 0 \rangle, \quad (1)$$

where  $j^\mu(x)$  is the electromagnetic current. It can be decomposed with respect to its Lorentz structure into two terms

$$\Pi^{\mu\nu}(q, M) = (-q^2 g^{\mu\nu} + q^\mu q^\nu) \Pi(q^2, M) + q^\mu q^\nu \Pi_L(q^2, M), \quad (2)$$

where  $\Pi(q^2, M)$  and  $\Pi_L(q^2, M)$  are the transversal and longitudinal parts of the vacuum polarization. The longitudinal part vanishes due to the Ward-Takahashi identities. In the following the symbols  $M_e$ ,  $M_\mu$ ,  $M_\tau$  denote the on-shell masses of the electron(e), muon( $\mu$ ) and tauon( $\tau$ ) with the mass hierarchy  $M_e \ll M_\mu \ll M_\tau$ . The polarization function  $\Pi(q^2, M)$  depends on the lepton masses  $M = \{M_e, M_\mu, M_\tau\}$  as well as on the external Minkowskian momentum  $q$ , where we consider here  $q^2 > 0$ . It is related to the effective electromagnetic coupling by

$$\alpha_{\text{eff}}(q^2) = \frac{\alpha}{1 + \text{Re}\Pi_{\text{OS}}(q^2)}, \quad (3)$$

which describes the running of the electromagnetic coupling to higher energy scales. The running from  $q^2 = 0$  to the scale of the electroweak  $Z$ -boson mass,  $q^2 = M_Z^2$ , is mediated by the parameter  $\Delta\alpha$ , where we consider within this work the purely leptonic contributions,

$$\Delta\alpha_{\text{lep}} \equiv \Delta\alpha_{\text{lep}}(M_Z^2) = -\text{Re}\Pi_{\text{OS}}(q^2 = M_Z^2), \quad (4)$$

from the Standard Model leptons which we defined above. The symbol  $\text{Re}$  denotes the real part of a complex quantity and the label  $\text{os}$  specifies that the vacuum polarization function has been renormalized in the on-shell scheme. In the on-shell scheme holds  $\Pi_{\text{OS}}(q^2 = 0) = 0$ .

In order to determine  $\Delta\alpha_{\text{lep}}$  we need to compute the vacuum polarization function in the on-shell scheme. We define its perturbative expansion as

$$\Pi_{\text{OS}}(q^2, M) = \frac{\alpha}{4\pi} \sum_{k=0} \left( \frac{\alpha}{\pi} \right)^k \Pi_{\text{OS}}^{(k)}(q^2, M), \quad (5)$$

where each term on the r.h.s. of Eq. (5) stands for the one-loop( $k=0$ ), two-loop( $k=1$ ), three-loop( $k=2$ ) and four-loop( $k=3$ ) contribution. The vacuum polarization function in the on-shell scheme at one- and two-loop order is known since long [10, 11]. The one-loop result reads

$$\Pi_{\text{OS}}^{(0)}(q^2, M) = \sum_{i=e,\mu,\tau} \frac{4}{3} \left[ \frac{5}{3} + x_i - (2 + x_i) \frac{\beta_i}{2} \log \left( \frac{\beta_i + 1}{\beta_i - 1} \right) \right] \quad (6)$$

$$q^2 \gg (2M_i)^2 \approx \sum_{i=e,\mu,\tau} \left[ -\frac{4}{3} L_{qM_i} + \frac{20}{9} + 8 \frac{M_i^2}{q^2} + 4 \left( \frac{M_i^2}{q^2} \right)^2 (2L_{qM_i} + 1) + \dots \right], \quad (7)$$

with  $\beta_i^2 = 1 - x_i$ ,  $x_i = (2M_i)^2/q^2$  and  $L_{qM_i} = \log(-q^2/M_i^2)$ . The ellipsis in Eq. (7) represent the contributions from higher orders in the expansion in  $M_i^2/q^2$  for an external momentum  $q^2$  which is much larger than the threshold for lepton pair production. The index  $i$  stands for the flavor of the lepton which is connected to the external photons. This notation will be kept also in the subsequent equations. The analytic continuation of the logarithm is determined by the  $i\epsilon$ -prescription.

The two-loop result is given by

$$\begin{aligned} \Pi_{\text{OS}}^{(1)}(q^2, M) = \sum_{i=e,\mu,\tau} & \left[ -L_{qM_i} + \frac{5}{6} - 4\zeta_3 - 12 \frac{M_i^2}{q^2} L_{qM_i} \right. \\ & \left. - \left( \frac{M_i^2}{q^2} \right)^2 \left( 12L_{qM_i}^2 + 10L_{qM_i} - \frac{2}{3} - 16\zeta_3 \right) + \mathcal{O}\left(\frac{M_i^6}{q^4}\right) \right]. \quad (8) \end{aligned}$$

The diagrams which contribute to the one- and two-loop vacuum polarization function are shown in Fig. 1.

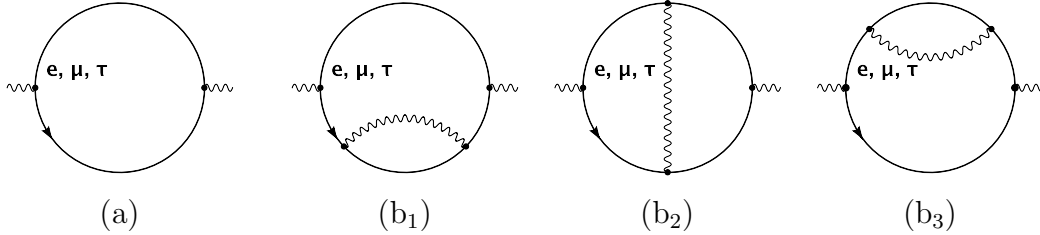


Figure 1: Diagrams contributing to the one-loop and two-loop vacuum polarization function. The solid lines denote charged leptons ( $e, \mu, \tau$ ); the wavy lines denote photons.

The corresponding result in QCD, where quarks instead of leptons are running in the fermion loop, can easily be deduce up to two-loop order from Eqs. (6) and (8) by multiplying them with the appropriate color factors.

The three-loop contribution has been determined in terms of expansions in the low ( $q^2 \rightarrow 0$ ) and the high energy limit in Refs. [13–21]. The result in the on-shell scheme has been computed in Ref. [12], where it has been decomposed into four contributions

$$\begin{aligned} \Pi_{\text{OS}}^{(2)}(q^2, M) = & \sum_{i=e,\mu,\tau} \left( \Pi_A^{(2)}(q^2, M_i) + \Pi_F^{(2)}(q^2, M_i) \right) \\ & + \sum_{\substack{i=\mu,\tau \\ j=e,\mu \\ j \neq i}} \Pi_l^{(2)}(q^2, M_i, M_j) + \sum_{\substack{i=e,\mu \\ j=\mu,\tau \\ j \neq i}} \Pi_h^{(2)}(q^2, M_i, M_j). \quad (9) \end{aligned}$$

The first term  $\Pi_A^{(2)}(q^2, M_i)$  is the quenched contribution which originates from diagrams without any insertion of an internal, closed fermion loop. In contrast, the terms  $\Pi_F^{(2)}(q^2, M_i)$ ,  $\Pi_l^{(2)}(q^2, M_i, M_j)$  and  $\Pi_h^{(2)}(q^2, M_i, M_j)$  arise from diagrams with internal lepton loop insertions. The contribution with the subscript  $F$  arises from diagrams with an internal lepton loop of the same flavor as the outer loop which is connected to the external photons; the contribution with the subscript  $h$  comes from diagrams with a heavier lepton inserted in the inner loop than in the outer one, and the subscript  $l$  denotes the contribution where the mass of the lepton in the inner loop is lighter than the external one. An example-diagram for each of these four different kind of contributions is shown in Fig. 2.

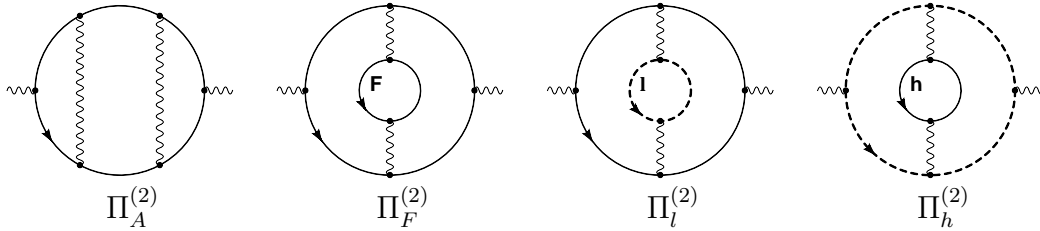


Figure 2: Examples for the different diagram classes at three-loop order. For each class one typical representative is shown. Loops of particles which are represented by a dashed line denote particles which have a lower mass than the ones which are represented by a solid line.

The four different contributions on the r.h.s. of Eq. (9) are renormalized in the on-shell scheme and we omit a label indicating it for brevity. The index  $i$  labels again the external leptons which connect to the external photons in a diagram, whereas the index  $j$  enumerates the internal leptons. The three-loop on-shell results for the different terms on the r.h.s. of Eq. (9) were determined in terms of expansions in the limit  $q^2 \gg (2M)^2$  in Ref. [12] and are shown for completeness in Appendix A.1, where we have in addition also added mass corrections of higher orders in the small ratio  $M^2/q^2$ . For  $\Pi_A^{(2)}(q^2, M_i)$  and  $\Pi_F^{(2)}(q^2, M_i)$  they can be obtained from the  $\overline{\text{MS}}$  results in Refs. [16, 17, 19, 21] after converting them from the  $\overline{\text{MS}}$  to the on-shell scheme.

The four-loop result can be decomposed, similarly to Eq. (9), into several gauge invariant contributions according to the properties and number of the inserted lepton loops

$$\begin{aligned}
\Pi_{\text{OS}}^{(3)}(q^2, M) &= \sum_{i=e,\mu,\tau} \left[ \Pi_A^{(3)}(q^2, M_i) + \Pi_{FF}^{(3)}(q^2, M_i) + \Pi_F^{(3)}(q^2, M_i) \right] \\
&+ \sum_{\substack{i=e,\mu \\ j=\mu,\tau \\ j \neq i}} \left[ \Pi_h^{(3)}(q^2, M_j) + \Pi_{Fh}^{(3)}(q^2, M_i, M_j) + \Pi_{hh}^{(3)}(q^2, M_j) \right] \\
&+ \sum_{\substack{i=\mu,\tau \\ j=e,\mu \\ j \neq i}} \left[ \Pi_l^{(3)}(q^2, M_i, M_j) + \Pi_{Fl}^{(3)}(q^2, M_i, M_j) + \Pi_{ll}^{(3)}(q^2, M_i, M_j) \right]
\end{aligned}$$

$$\begin{aligned}
& + \Pi_{e\mu}^{(3)}(q^2, M_\tau, M_e, M_\mu) + \sum_{j=e,\mu} \Pi_{j\tau}^{(3)}(q^2, M_j, M_\tau) \\
& + \sum_{i=e,\mu,\tau} \Pi_{s,d}^{(3)}(q^2, M_i) + \sum_{\substack{k,l=e,\mu,\tau \\ k \neq l}} \Pi_{s,nd}^{(3)}(q^2, \max\{M_k, M_l\}) . \tag{10}
\end{aligned}$$

The property which characterizes each individual term on the r.h.s. of Eq. (10) is defined in Fig. 3 by the shown example-diagrams. Singlet diagrams, in which the external photons are not connected to the same fermion loop did not yet contribute at three-loop order due to Furry's theorem [22] and survive for the first time at four-loop order. They are given by the terms  $\Pi_{s,d}^{(3)}(q^2, M_i)$  and  $\Pi_{s,nd}^{(3)}(q^2, \max\{M_k, M_l\})$ , where the subscript  $d$  stands for ‘diagonal’ which denotes diagrams with the same lepton flavor in the two fermion loops, whereas the subscript  $nd$  stands for ‘non-diagonal’ and denotes diagrams with different lepton flavors in the two fermion loops. The diagrams with three different lepton flavors which correspond to the three terms  $\Pi_j^{(3)}$  with  $j = \{e\mu, e\tau, \mu\tau\}$  of Eq. (10), also arise for the first time at four-loop order. As in the previous Eq. (9) the subscripts  $\{FF, Fh, Fh, hh, l, Fl, ll\}$  of the non-singlet contributions in Eq. (10) refer always to the internal leptons. Apart from that we show only the function arguments of the lepton masses on which these expanded results will depend to the considered expansion depth. In general, without expansion, the functions may depend on more mass arguments than those which are given here. The on-shell results for  $\Pi_A^{(3)}(q^2, M_i)$ ,  $\Pi_{FF}^{(3)}(q^2, M_i)$ ,  $\Pi_F^{(3)}(q^2, M_i)$  and  $\Pi_{s,d}^{(3)}(q^2, M_i)$  have already been determined in Refs. [23, 24] in the high energy expansion. They are given for completeness in Appendix A.2. The calculation of the terms in Eq. (10) which are still unknown in order to describe the complete lepton mass hierarchy will be discussed in the next section.

### 3 Calculation

In order to compute the leading four-loop QED corrections to  $\Delta\alpha_{\text{lep}}$  we will determine the on-shell vacuum polarization function contribution  $\Pi_{\text{OS}}^{(3)}(q^2 = M_Z^2, M)$  of Eq. (10) in the small lepton mass expansion up to power corrections of the order  $M^2/M_Z^2$ . For this purpose we need to know the  $\overline{\text{MS}}$  vacuum polarization function on the one hand for massless leptons and on the other hand we need to know it in the low energy limit at  $q^2 = 0$  for the renormalization procedure in the on-shell scheme as we will see in the following.

The vacuum polarization function in the massless limit,  $\overline{\Pi}^{(k)}(q^2, \overline{m} = 0)$ , is known analytically in the  $\overline{\text{MS}}$  scheme up to four-loop order ( $k = 3$ ) [13, 14, 20, 23–27]. In this limit it does not depend on the lepton masses but only on the external momentum. The perturbative expansion of the vacuum polarization function in the  $\overline{\text{MS}}$  scheme is defined here in complete analogy to the on-shell case shown in Eq. (5), only that all quantities are renormalized in the  $\overline{\text{MS}}$  scheme.

The non-singlet contributions of the vacuum polarization function in the  $\overline{\text{MS}}$  scheme at four-loop order in QCD and at  $q^2 = 0$  were computed in Ref. [28]. The QED result

was presented in Refs. [23, 24]. Higher moments of the low energy expansion in  $q^2/\overline{m}^2$  were determined in Refs. [28–34] at four-loop order. In these works the contributions from quenched diagrams were determined as well as the contributions from diagrams with inserted fermion loops, where the internal fermions were massless or had the same mass as the external one. In order to perform the determination of  $\Delta\alpha_{\text{lep}}$  in the previously described expansion at four-loop order in QED one needs in addition still to compute the contributions at  $q^2 = 0$  from diagrams where the internal fermion loops have a larger mass than the external one as well as the non-diagonal singlet contributions. In Fig. 3 we show one four-loop example-diagram for each arising gauge invariant subset of diagrams. These gauge invariant subsets are given by the different terms in Eq. (10).

The two ingredients, the vacuum polarization function in the  $\overline{\text{MS}}$  scheme in the massless limit and at  $q^2 = 0$ , are then combined to construct the vacuum polarization function in the on-shell scheme

$$\Pi_{\text{Os}}(q^2, \alpha, M) = \frac{\alpha}{4\pi} \sum_{k=0} \left(\frac{\bar{\alpha}}{\pi}\right)^k \left[ \overline{\Pi}^{(k)}(q^2, \overline{m} = 0) - \overline{\Pi}^{(k)}(q^2 = 0, \overline{m}) \right] \Big|_{\substack{\overline{m}=C_{\overline{m}M}M \\ \bar{\alpha}=C_{\bar{\alpha}\alpha}\alpha}} + \mathcal{O}\left(\frac{M^2}{q^2}\right), \quad (11)$$

where the ‘bar’ on the r.h.s. indicates that a quantity has been renormalized in the  $\overline{\text{MS}}$  scheme, e.g.  $\bar{\alpha}$  is the fine-structure constant in the  $\overline{\text{MS}}$  scheme. Masses in the  $\overline{\text{MS}}$  scheme are denoted by small letters  $\overline{m}$ , whereas masses in the on-shell scheme are given by capital letters  $M$ . The conversion function  $C_{\overline{m}M}(C_{\bar{\alpha}\alpha})$  converts the lepton mass(fine-structure constant) from the on-shell to the  $\overline{\text{MS}}$  scheme or vice versa.

In order to compute  $\overline{\Pi}^{(3)}(q^2 = 0, \overline{m})$  at four-loop order in QED we start first to generate the required diagrams with the program **QGRAF** [35]. The expansion around  $q^2 = 0$  is performed with the programs **q2e** and **exp** [36–38] which also map the resulting loop integrals onto a proper notation, which allows us to perform further simplifications of the resulting loop integrals in a straightforward way. In the next step the integrals are reduced to a small set of master integrals with the traditional integration-by-parts (IBP) method in combination with Laporta’s algorithm [39, 40]. For the algebraic manipulations we use **FORM** [41–43] and the rational functions in the space-time dimension  $d$ , which arise while solving the linear system of IBP equations, are simplified with **FERMAT** [44]. At the end of this procedure all four-loop integrals can be expressed in terms of 12 master integrals from which three are factorized. They are known since long in the literature [45, 46]; also other authors have contributed to individual master integrals or specific orders in the  $\varepsilon = 2 - d/2$  expansion with analytic results [29, 47–53] which allow to obtain  $\overline{\Pi}^{(3)}(q^2 = 0, \overline{m})$  in Eq. (11) completely analytically.

To check our calculation we have kept the dependence on the gauge parameter of the photon propagator and have verified its cancellation in the final result. We have also checked the Ward-Takahashi identity and verified that the longitudinal part of the vacuum polarization function vanishes.

For the conversion of the vacuum polarization function to the on-shell scheme in Eq. (11) the conversion functions  $C_{\overline{m}M}$  for the lepton mass and  $C_{\bar{\alpha}\alpha}$  for the fine-structure constant are needed.

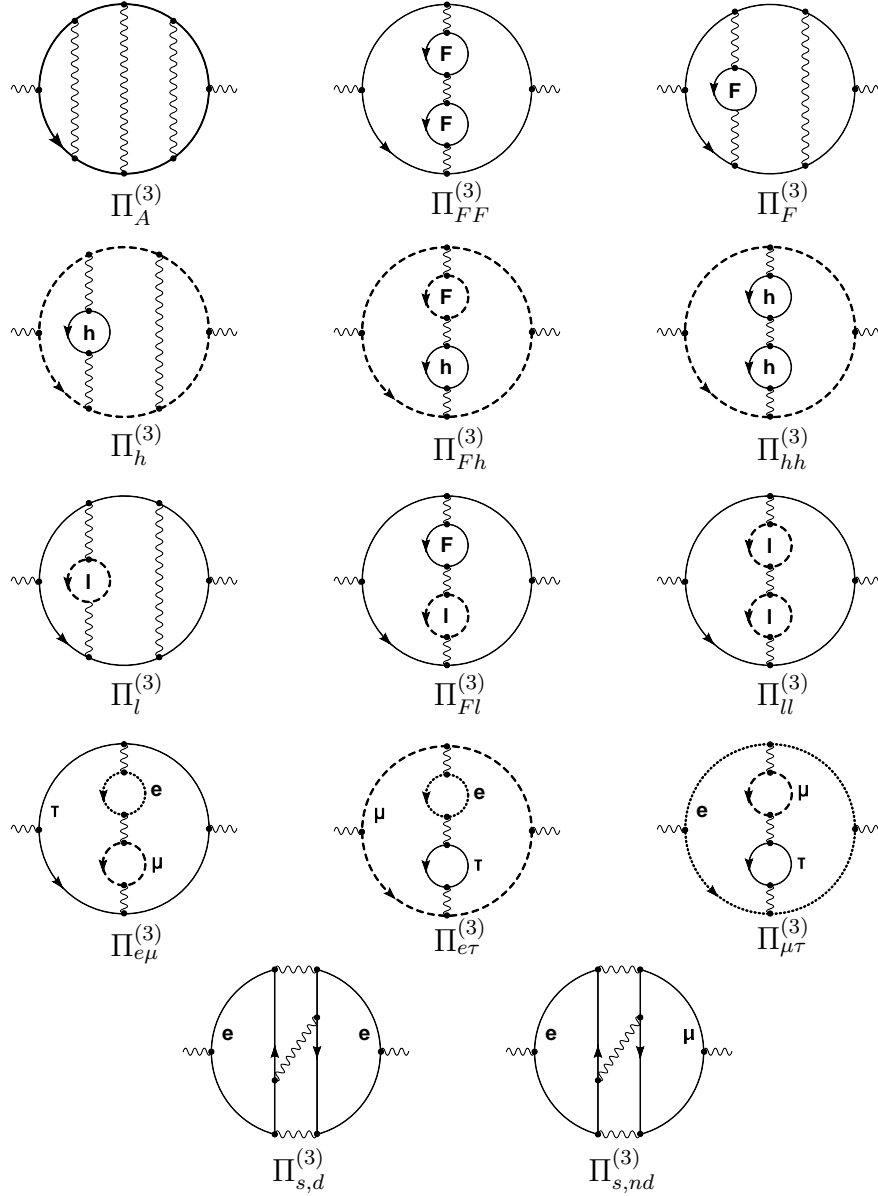


Figure 3: For each of the different terms in Eq. (10) is shown here one typical four-loop diagram as representative of the whole class. The wavy lines denote photons; the lines with an arrow are leptons. Dashed lines represent particles which have a smaller mass than particles which are given by solid lines. Dotted lines denote particles which have a smaller mass than particles which are represented by dashed lines. The singlet type diagrams ( $\Pi_{s,d}^{(3)}$ ,  $\Pi_{s,nd}^{(3)}$ ) contribute for the first time at four-loop order in QED.

Let us start with the conversion of the lepton masses. For the case of the mass of the  $\tau$ -lepton the QED  $\overline{\text{MS}}$ -on-shell relation is known to three-loop order; it can be deduced from the QCD results of Refs. [54–60]. We decompose the different orders in perturbation



theory as follows

$$\overline{m}_\ell = M_\ell \underbrace{\left[ 1 + \left( \frac{\bar{\alpha}}{\pi} \right) c_\ell^{(1)} + \left( \frac{\bar{\alpha}}{\pi} \right)^2 c_\ell^{(2)} + \left( \frac{\bar{\alpha}}{\pi} \right)^3 c_\ell^{(3)} + \mathcal{O}(\bar{\alpha}^4) \right]}_{=C_{\overline{m}M}}. \quad (12)$$

The coefficient functions  $c_\ell^{(k)}$  ( $k = 1, 2, 3$ ) depend in general on the renormalization scale  $\mu$  and on the lepton masses, where we have omitted these function arguments for simplicity. The one-loop coefficient is known since long [54] and given in Eq. (47). Starting from two-loop order also internal lepton loops can arise, like shown in Fig. 4. In general, the coefficient  $c_\ell^{(2)}$  can be decomposed into gauge invariant contributions which arise from quenched ( $A$ ) diagrams, contributions from diagrams with light ( $l$ ) or heavy ( $h$ ) inserted lepton loops and diagrams where the internal lepton loop has the same mass as the external one ( $F$ ).

The tauon is the heaviest lepton of the SM, so that only light internal leptons or the tauon itself can contribute in its conversion relation of Eq. (12). At two-loop order one obtains

$$c_\tau^{(2)}(M_\tau) = c_A^{(2)}(M_\tau) + 2c_l^{(2)}(M_\tau) + c_F^{(2)}(M_\tau). \quad (13)$$

The individual terms on the r.h.s. of Eq. (13) are known [55] and are given in Appendix C.

At three-loop order the same decomposition with respect to the number of inserted closed lepton loops can be made

$$c_\tau^{(3)}(M_\tau) = c_A^{(3)}(M_\tau) + 2c_l^{(3)}(M_\tau) + c_F^{(3)}(M_\tau) + 2c_{Fl}^{(3)}(M_\tau) + 4c_{ll}^{(3)}(M_\tau) + c_{FF}^{(3)}(M_\tau), \quad (14)$$

where the last three terms with two letters in the subscript arise from diagrams with two lepton loop insertions. The coefficients on the r.h.s. of Eq. (14) are again known [56, 57, 59–61] and given for completeness in Appendix C. In particular the complete effects from a second virtual massive fermion in the inner loops were determined at two- and three-loop order in Refs. [55, 60]. Since our final result for  $\Delta\alpha_{\text{lep}}$  at four-loop order will be in an expansion in  $M^2/M_Z^2$  up to the constant order, we do not consider higher order virtual mass effects in Eqs. (13) and (14) as well as in the following Eqs. (15), (16), (19) and (20).

For the conversion relation of the muon internal heavy  $\tau$ -lepton loops can arise starting from two-loop order. These internal  $\tau$ -lepton contributions have been determined in Ref. [12] and lead to the additional contribution  $c_h^{(2)}(M_\mu, M_\tau)$

$$c_\mu^{(2)}(M_\mu, M_\tau) = c_A^{(2)}(M_\mu) + c_l^{(2)}(M_\mu) + c_F^{(2)}(M_\mu) + c_h^{(2)}(M_\mu, M_\tau). \quad (15)$$

Similarly also at three-loop order the new contributions with heavy fermion loop insertions arise

$$c_\mu^{(3)}(M_\mu, M_\tau) = c_A^{(3)}(M_\mu) + c_l^{(3)}(M_\mu) + c_F^{(3)}(M_\mu) + c_h^{(3)}(M_\mu, M_\tau)$$

$$\begin{aligned}
& + c_{Fl}^{(3)}(M_\mu) + c_{lh}^{(3)}(M_\mu, M_\tau) + c_{Fh}^{(3)}(M_\mu, M_\tau) \\
& + c_{ll}^{(3)}(M_\mu) + c_{FF}^{(3)}(M_\mu) + c_{hh}^{(3)}(M_\mu, M_\tau).
\end{aligned} \tag{16}$$

The coefficient functions  $c_h^{(3)}(M_\mu, M_\tau)$ ,  $c_{hh}^{(3)}(M_\mu, M_\tau)$  and  $c_{Fh}^{(3)}(M_\mu, M_\tau)$  will be presented in Section 4 and have been determined with the help of the decoupling relations for the fine-structure constant and the fermion mass

$$\bar{\alpha}^{(n_f-1)}(\mu) = \zeta_\gamma^2(\mu, \bar{\alpha}^{(n_f)}(\mu), \bar{m}_h) \bar{\alpha}^{(n_f)}(\mu), \tag{17}$$

$$\bar{m}^{(n_f-1)}(\mu) = \zeta_m(\mu, \bar{\alpha}^{(n_f)}(\mu), \bar{m}_h) \bar{m}^{(n_f)}(\mu). \tag{18}$$

The decoupling functions  $\zeta_\gamma^2(\mu, \bar{\alpha}^{(n_f)}(\mu), \bar{m}_h)$  and  $\zeta_m(\mu, \bar{\alpha}^{(n_f)}(\mu), \bar{m}_h)$  can be obtained from the known QCD results [62–69] and are given in Appendix B for completeness. The mass  $\bar{m}_h$  is the mass of the decoupled heavy fermion and the superscript  $n_f$  denotes the number of active lepton flavors. As a check of this procedure we reproduced in Eq. (15) the known two-loop result of Ref. [12] for  $c_h^{(2)}(M_\mu, M_\tau)$ . It can also be obtained from the results in Refs. [55,60], which incorporate the complete fermion mass dependence. In the same way we derived also  $c_{lh}^{(3)}(M_\mu, M_\tau)$  and compared it with the analytic results of Ref. [60]. The remaining coefficient functions in Eq. (16) are again known and given in Appendix C.

In the case of the mass conversion relation from the  $\overline{\text{MS}}$  to the on-shell scheme for the electron the internal loops in the Feynman diagrams cannot have a lower mass than the external fermion, since only internal electrons or heavy muons and  $\tau$ -leptons can appear. These situations are shown in Fig. 4.

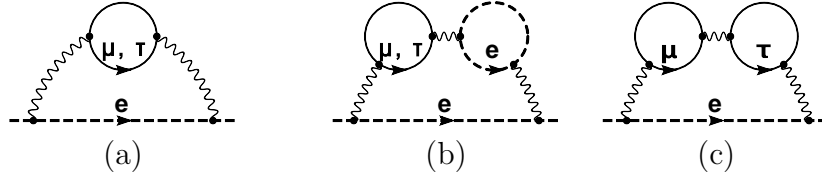


Figure 4: Lepton self-energy example-diagrams at two-loop and three-loop order in QED which contribute to Eq. (12). The wavy lines are photons, the dashed and solid lines are leptons. Leptons with a dashed line have a smaller mass than leptons with a solid line.

Similarly to Eqs. (13) and (15) the two-loop contribution reads

$$c_e^{(2)}(M_e, M_\mu, M_\tau) = c_A^{(2)}(M_e) + c_F^{(2)}(M_e) + c_h^{(2)}(M_e, M_\mu) + c_h^{(2)}(M_e, M_\tau) \tag{19}$$

and the three-loop term follows in analogy to Eqs. (14) and (16)

$$\begin{aligned}
c_e^{(3)}(M_e, M_\mu, M_\tau) &= c_A^{(3)}(M_e) + c_F^{(3)}(M_e) + c_h^{(3)}(M_e, M_\mu) + c_h^{(3)}(M_e, M_\tau) \\
&+ c_{Fh}^{(3)}(M_e, M_\mu) + c_{Fh}^{(3)}(M_e, M_\tau) + c_{\mu\tau}^{(3)}(M_e, M_\mu, M_\tau) \\
&+ c_{FF}^{(3)}(M_e) + c_{hh}^{(3)}(M_e, M_\mu) + c_{hh}^{(3)}(M_e, M_\tau).
\end{aligned} \tag{20}$$

The new contributions with heavy fermion loop insertions are the terms with the coefficient functions which have a subscript  $h$  as well as the term  $c_{\mu\tau}^{(3)}(M_e, M_\mu, M_\tau)$ . The latter arises

from diagrams like the one shown in Fig. 4(c). These contributions have again been obtained with the decoupling relations of Eqs. (17) and (18) and are given in the next section.

We have checked that the dependence of the conversion relations on the renormalization scale  $\mu$  obeys the mass anomalous dimension [54, 70–72].

As last ingredient the conversion factor  $C_{\bar{\alpha}\alpha}$  which is mediating between the fine-structure constant renormalized in the  $\overline{\text{MS}}$  and the on-shell scheme,  $\bar{\alpha}(\mu) = C_{\bar{\alpha}\alpha}\alpha$ , is needed to order  $\alpha^2$ . It is known in literature [23, 24, 47] to four-loop order and can be obtained with the help of the concept of the invariant charge [73, 74] by computing the vacuum polarization function at  $q^2 = 0$  in the  $\overline{\text{MS}}$  scheme. The conversion factor respecting the mass hierarchy of the three charged leptons reads

$$C_{\bar{\alpha}\alpha} = 1 + \frac{\alpha}{\pi} \sum_{j=e,\mu,\tau} \frac{L_{\mu M_j}}{3} + \left(\frac{\alpha}{\pi}\right)^2 \sum_{j=e,\mu,\tau} \left[ \frac{15}{16} + \frac{L_{\mu M_j}}{4} + \sum_{k=e,\mu,\tau} \frac{L_{\mu M_j}}{3} \frac{L_{\mu M_k}}{3} \right] + \mathcal{O}(\alpha^3). \quad (21)$$

The symbol  $L_{\mu M}$  denotes the logarithm  $L_{\mu M} = \log(\mu^2/M^2)$ .

We have checked by combining the result of the vacuum polarization function in the  $\overline{\text{MS}}$  scheme up to four-loop order at  $q^2 = 0$  with the above  $\overline{\text{MS}}$  to on-shell conversion relations for the lepton masses and the fine-structure constant that  $\Pi_{\text{OS}}(q^2 = 0)$  is zero. We have also checked that the renormalization scale dependence  $\mu$  of the vacuum polarization function cancels after the conversion to the on-shell scheme.

## 4 Results

In this section we summarize our results obtained with the methods described in the previous section. We begin with the terms due to heavier leptons in the conversion relation of the lepton masses between the  $\overline{\text{MS}}$  and the on-shell scheme at three-loop order. The results which are known in literature are listed for completeness in Appendix C. The three terms which contribute in Eqs. (16) and (20) read

$$\begin{aligned} c_h^{(3)}(M_\ell, M_x) &= \frac{547}{192} + \frac{11}{360}\pi^4 + \frac{\pi^2}{6}\log^2(2) - \frac{1}{6}\log^4(2) - \frac{57}{32}\zeta_3 - 4a_4 \\ &\quad + \frac{359}{384}L_{\mu M_\ell} + L_{\mu M_x} \left[ \frac{1}{48} + \frac{5}{24}\pi^2 - \frac{\pi^2}{3}\log(2) \right. \\ &\quad \left. - \frac{1}{4}\zeta_3 - \frac{1}{16}L_{\mu M_\ell} \left( \frac{13}{2} + 3L_{\mu M_\ell} - \frac{3}{2}L_{\mu M_x} \right) \right], \end{aligned} \quad (22)$$

$$\begin{aligned} c_{hh}^{(3)}(M_\ell, M_x) &= \frac{1685}{7776} - \frac{7}{18}\zeta_3 + L_{\mu M_x} \left[ \frac{31}{108} \right. \\ &\quad \left. - L_{\mu M_x} \left( \frac{13}{72} + \frac{1}{12}L_{\mu M_\ell} - \frac{1}{18}L_{\mu M_x} \right) \right], \end{aligned} \quad (23)$$

$$c_{Fh}^{(3)}(M_\ell, M_x) = -\frac{1327}{3888} + \frac{2}{9}\zeta_3 - L_{\mu M_x} \left[ \frac{5}{8} - \frac{\pi^2}{9} \right]$$

$$+ \frac{1}{12} L_{\mu M_\ell} \left( \frac{13}{3} + L_{\mu M_\ell} \right) - \frac{1}{36} L_{\mu M_x}^2 \Big] , \quad (24)$$

$$\begin{aligned} c_{\mu\tau}^{(3)}(M_e, M_\mu, M_\tau) = & -\frac{1327}{3888} + \frac{2}{9}\zeta_3 + \frac{1}{6} L_{\mu M_\tau} \left[ \frac{31}{9} + \frac{1}{6} L_{\mu M_\tau}^2 \right. \\ & \left. - L_{\mu M_\mu} \left( \frac{13}{6} + L_{\mu M_e} - \frac{1}{2} L_{\mu M_\mu} \right) \right] , \end{aligned} \quad (25)$$

with  $L_{\mu M} = \log\left(\frac{\mu^2}{M^2}\right)$ . Here we defined  $a_n = \text{Li}_n(\frac{1}{2})$  with the polylogarithm function  $\text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$ . The symbol  $\zeta_n$  is the Riemann zeta function  $\zeta_n = \sum_{k=1}^{\infty} k^{-n}$ . The lepton mass  $M_x$  in Eqs. (22)-(24) is always larger than the lepton mass  $M_\ell$ .

With these results one can derive the leptonic contributions to the on-shell vacuum polarization function from all three charged lepton generations at four-loop order in QED in an expansion for large external momentum. As a check of our computer routines which perform the algebraic manipulations we reproduce the known one-loop and two-loop results as well as the three-loop results  $\Pi_A^{(2)}(q^2, M_i)$ ,  $\Pi_l^{(2)}(q^2, M_i, M_j)$ ,  $\Pi_F^{(2)}(q^2, M_i)$ ,  $\Pi_h^{(2)}(q^2, M_i, M_j)$  which are given in Eqs. (5)-(10) of Ref. [12] and list them for completeness in Appendix A. The new four-loop results in Eq. (10) read

$$\begin{aligned} \Pi_h^{(3)}(q^2, M_j) = & -\frac{1}{12} L_{qM_j}^2 - L_{qM_j} \left( \frac{1}{3} + \frac{19}{3}\zeta_3 - \frac{20}{3}\zeta_5 \right) + \frac{133}{24} \\ & + \frac{5}{108}\pi^4 + \frac{1657}{72}\zeta_3 - 4\zeta_3^2 - \frac{250}{9}\zeta_5 + \frac{2}{9}\pi^2 \log^2(2) \\ & - \frac{2}{9}\log^4(2) - \frac{16}{3}a_4 + \dots, \end{aligned} \quad (26)$$

$$\begin{aligned} \Pi_l^{(3)}(q^2, M_i, M_j) = & \frac{1}{12} L_{qM_i}^2 - \frac{1}{6} L_{qM_j} L_{qM_i} + L_{qM_i} \left( \frac{41}{36} + \frac{5}{9}\pi^2 - \frac{8}{9}\pi^2 \log(2) \right. \\ & \left. - \frac{29}{24}\zeta_3 \right) - L_{qM_j} \left( \frac{53}{36} + \frac{5}{9}\pi^2 - \frac{8}{9}\pi^2 \log(2) + \frac{41}{8}\zeta_3 - \frac{20}{3}\zeta_5 \right) \\ & + \frac{553}{108} + \frac{107}{54}\pi^2 - \frac{919}{12960}\pi^4 + \frac{4117}{144}\zeta_3 - 4\zeta_3^2 - \frac{250}{9}\zeta_5 \\ & + \frac{7}{108}\log^4(2) + \frac{89}{108}\pi^2 \log^2(2) - \frac{88}{27}\log(2)\pi^2 + \frac{14}{9}a_4 + \dots, \end{aligned} \quad (27)$$

$$\begin{aligned} \Pi_{Fh}^{(3)}(q^2, M_i, M_j) = & \frac{1}{27} L_{qM_j}^3 - \frac{1}{9} L_{qM_i} L_{qM_j}^2 + L_{qM_i} L_{qM_j} \left( \frac{11}{9} - \frac{8}{9}\zeta_3 \right) \\ & - L_{qM_j} \left( \frac{271}{81} - \frac{76}{27}\zeta_3 \right) - L_{qM_i} \left( \frac{37}{9} - \frac{76}{27}\zeta_3 \right) + \frac{3956}{243} \\ & - \frac{776}{81}\zeta_3 - \frac{40}{9}\zeta_5 + \dots, \end{aligned} \quad (28)$$

$$\begin{aligned} \Pi_{Fl}^{(3)}(q^2, M_i, M_j) = & \frac{1}{27} L_{qM_i}^3 - \frac{1}{9} L_{qM_j} L_{qM_i}^2 + L_{qM_j} L_{qM_i} \left( \frac{11}{9} - \frac{8}{9}\zeta_3 \right) \\ & - L_{qM_i} \left( \frac{703}{108} - \frac{8}{27}\pi^2 - \frac{671}{216}\zeta_3 \right) - L_{qM_j} \left( \frac{307}{324} + \frac{8}{27}\pi^2 - \frac{545}{216}\zeta_3 \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{23689}{1944} + \frac{26}{81}\pi^2 - \frac{49}{4320}\pi^4 - \frac{2663}{324}\zeta_3 - \frac{40}{9}\zeta_5 \\
& - \frac{1}{36}\pi^2 \log^2(2) + \frac{1}{36}\log^4(2) + \frac{2}{3}a_4 + \dots,
\end{aligned} \tag{29}$$

$$\begin{aligned}
\Pi_{hh}^{(3)}(q^2, M_j) &= -\frac{1}{27}L_{qM_j}^3 + L_{qM_j}^2 \left( \frac{11}{18} - \frac{4}{9}\zeta_3 \right) - L_{qM_j} \left( \frac{302}{81} - \frac{76}{27}\zeta_3 \right) \\
& + \frac{4207}{486} - \frac{442}{81}\zeta_3 - \frac{20}{9}\zeta_5 + \dots,
\end{aligned} \tag{30}$$

$$\begin{aligned}
\Pi_{ll}^{(3)}(q^2, M_i, M_j) &= -\frac{1}{27}L_{qM_i}^3 + \frac{1}{9}L_{qM_i}L_{qM_j} (L_{qM_i} - L_{qM_j}) - \frac{14}{27}L_{qM_i}^2 \\
& + L_{qM_j}^2 \left( \frac{5}{54} - \frac{4}{9}\zeta_3 \right) + \frac{28}{27}L_{qM_j}L_{qM_i} - L_{qM_i} \left( \frac{70}{81} + \frac{4}{27}\pi^2 \right) \\
& - L_{qM_j} \left( \frac{232}{81} - \frac{4}{27}\pi^2 - \frac{76}{27}\zeta_3 \right) + \frac{3329}{486} - \frac{26}{81}\pi^2 \\
& - \frac{442}{81}\zeta_3 - \frac{20}{9}\zeta_5 + \dots,
\end{aligned} \tag{31}$$

$$\begin{aligned}
\Pi_{e\mu}^{(3)}(q^2, M_\tau, M_e, M_\mu) &= -\frac{2}{27}L_{qM_\tau}^3 - \frac{2}{9}L_{qM_e}L_{qM_\mu}L_{qM_\tau} + \frac{1}{9}L_{qM_\tau}^2 (L_{qM_e} + L_{qM_\mu}) \\
& - \frac{28}{27}L_{qM_\tau}^2 + \frac{28}{27}L_{qM_\tau} (L_{qM_e} + L_{qM_\mu}) + L_{qM_e}L_{qM_\mu} \left( \frac{5}{27} - \frac{8}{9}\zeta_3 \right) \\
& - (L_{qM_e} + L_{qM_\mu}) \left( \frac{232}{81} - \frac{4}{27}\pi^2 - \frac{76}{27}\zeta_3 \right) - L_{qM_\tau} \left( \frac{140}{81} \right. \\
& \left. + \frac{8}{27}\pi^2 \right) + \frac{3329}{243} - \frac{52}{81}\pi^2 - \frac{884}{81}\zeta_3 - \frac{40}{9}\zeta_5 + \dots,
\end{aligned} \tag{32}$$

$$\begin{aligned}
\Pi_{j\tau}^{(3)}(q^2, M_j, M_\tau) &= \frac{1}{27}L_{qM_\tau}^3 - \frac{1}{9}L_{qM_\tau}^2 L_{qM_j} + L_{qM_j}L_{qM_\tau} \left( \frac{11}{9} - \frac{8}{9}\zeta_3 \right) \\
& - L_{qM_\tau} \left( \frac{271}{81} - \frac{76}{27}\zeta_3 \right) - L_{qM_j} \left( \frac{37}{9} - \frac{76}{27}\zeta_3 \right) + \frac{3956}{243} \\
& - \frac{776}{81}\zeta_3 - \frac{40}{9}\zeta_5 + \dots, \quad j = \{e, \mu\}
\end{aligned} \tag{33}$$

$$\Pi_{s,nd}^{(3)}(q^2, M) = -L_{qM} \left( \frac{11}{9} - \frac{8}{3}\zeta_3 \right) + \frac{82}{27} - \frac{80}{9}\zeta_3 - \frac{8}{3}\zeta_3^2 + \frac{20}{3}\zeta_5 + \dots, \tag{34}$$

with the logarithm  $L_{qM} = \log(-q^2/M^2)$ . The dots stand for a deeper expansion in the ratio of the lepton masses and the external momentum squared,  $M^2/q^2$ . The remaining symbols are the same as those defined after Eqs. (22)-(25).

In the following we use Eq. (4) to perform a numerical evaluation of  $\Delta\alpha_{\text{lep}}$  by using the available results of the vacuum polarization function in the on-shell scheme up to four-loop order in order to study the relative size of the different contributions. For the numerical evaluation we use the following input values from CODATA [1] and PDG [75] for the lepton masses  $M_e = 0.510998928(11)$  MeV,  $M_\mu = 105.6583715(35)$  MeV,  $M_\tau = 1776.82(16)$  MeV, and for the  $Z$ -boson mass  $M_Z = 91.1876(21)$  GeV. With these values

the coefficients of the different orders in perturbation theory of  $\Delta\alpha_{\text{lep}}$  read

$$\begin{aligned}\Delta\alpha_{\text{lep}} &= \left(\frac{\alpha}{\pi}\right) 13.52631(8) + \left(\frac{\alpha}{\pi}\right)^2 14.38553(6) \\ &+ \left(\frac{\alpha}{\pi}\right)^3 84.8285(7) + \left(\frac{\alpha}{\pi}\right)^4 [810.65(1)_{\text{NS}} - 39.8893(5)_{\text{SI}}] + \mathcal{O}(\alpha^5),\end{aligned}\quad (35)$$

where we have decomposed the new four-loop term into the contribution from non-singlet(NS) and singlet(SI) type of diagrams. The size of the contribution from the singlet diagrams is much smaller than the one from non-singlet diagrams. For the numerical evaluations in this section we use for the one-loop contribution the unexpanded result of Eq. (6). At two- and three-loop order we take the expanded results, where we include as many series coefficients in the small lepton mass expansion until their contribution becomes negligible compared to the numerical values given here. The same holds for the corresponding values in Table 1. Some of the lowest expansion coefficients are shown in Eq. (8) for the two-loop order and in Eqs. (37)-(40) for the three-loop contribution. At four loops we use the new result up to the constant order in the mass expansion which was presented in this section in combination with the known terms which are summarized in Appendix A.2.

Evaluating also numerically the fine-structure constant  $\alpha = 7.2973525698(24) \times 10^{-3}$ , which is taken from CODATA [1], one obtains

$$\Delta\alpha_{\text{lep}} = 0.0314192\dots + 0.00007762\dots + 0.00000106\dots + 0.00000002\dots, \quad (36)$$

where each term stands for the next order in the perturbative expansion of QED, i.e. the one-loop, two-loop, three-loop and four-loop contribution.

The power suppressed terms (proportional to  $M^2/M_Z^2$  and higher powers) at two-loop order amount to about  $\sim 5 \times 10^{-8}$  and are smaller than the three-loop result up to the constant order in the small mass expansion. The latter amounts to about  $\sim 10^{-6}$ . The power suppressed terms at three-loop order amount to about  $\sim 4 \times 10^{-10}$  and are again smaller than the four-loop result up to the constant order, which is  $\sim 2 \times 10^{-8}$ .

A more detailed decomposition of the individual contributions which arise from different gauge invariant subsets of diagrams is listed in Table 1 for the non-singlet contributions and in Table 2 for the four-loop singlet contributions.

At three-loop and also at four-loop order the dominant contribution to  $\Delta\alpha_{\text{lep}}$  emerges from diagrams with internal lepton loop insertions, especially from the insertions of electrons, which can be seen in Table 1. The reason for this dominance is an logarithmic enhancement of these terms which arises from large logarithms in the small mass ratio  $M_e^2/M_Z^2$ . In comparison the contributions from quenched diagrams are small. This behavior was already observed at three-loop order in Ref. [12] and also holds at four-loop order.

Summing up all contributions one obtains  $\Delta\alpha_{\text{lep}} = 0.0314979$  with an uncertainty of about  $\sim 2 \cdot 10^{-7}$ , mainly due to the error in the  $\tau$ -lepton mass and the  $Z$ -boson mass. This uncertainty is about one order of magnitude bigger than the size of the new four-loop contribution, which is about  $\sim 2 \cdot 10^{-8}$ , so that the theory uncertainty in  $\Delta\alpha_{\text{lep}}$  due

$10^7 \times \Delta\alpha_{\text{lep}}$		outer leptons			outer leptons $e + \mu + \tau$
		$e$	$\mu$	$\tau$	
1-loop		174346.6(4)	91784.3(4)	48061(1)	314192(2)
2-loop		379.8293(6)	235.9993(6)	160.341(2)	776.170(3)
3-loop	quenched	-0.1356831(2)	-0.0939860(2)	-0.077406200(3)	-0.3070753(4)
inner leptons	$\left\{ \begin{array}{l} e \\ \mu \\ \tau \end{array} \right.$	2.86735(1)	2.66078(1)	2.14729(2)	7.67542(4)
		0.839626(6)	0.844870(6)	0.823473(6)	2.50797(2)
		0.24972(1)	0.24972(1)	0.25560(1)	0.75503(4)
	sum	3.82101(2)	3.66138(2)	3.14895(4)	10.63134(9)
4-loop	quenched	-0.00069	-0.00013	0.00016	-0.00066
inner leptons	$\left\{ \begin{array}{l} e \\ \mu \\ \tau \end{array} \right.$	0.00512	0.00391	0.00261	0.01164
		0.00197	0.00188	0.00150	0.00535
		0.00081	0.00081	0.00072	0.00234
	$ee$	0.03361	0.03318	0.02955	0.09634
	$\mu\mu$	0.00497	0.00498	0.00509	0.01504
	$\tau\tau$	0.00061	0.00061	0.00062	0.00183
	$e\mu$	0.02376	0.02386	0.02375	0.07137
	$e\tau$	0.00748	0.00748	0.00763	0.02259
	$\mu\tau$	0.00336	0.00336	0.00342	0.01014
	sum	0.08100	0.07994	0.07505	0.23599
1+2+3+4-loop		174730.4(4)	92024.0(4)	48225(1)	314979(2)

Table 1: The table shows the numerical evaluation of the different gauge invariant subsets which contribute to  $\Delta\alpha_{\text{lep}}$  from one-loop to four-loop order. Some example-diagrams which correspond to the different contributions are given in Figs. 1, 2 and 3. Starting from three-loop order the table shows the different contributions which arise from diagrams with external(outer) leptons and internal(inner) leptons separately. At four-loop order arise for the first time diagrams with two internal lepton insertions. The non-singlet contributions at four-loop order are not given here, but are shown in Table 2.

$10^7 \times \Delta\alpha_{\text{lep}}$		leptons in right loop		
		$e$	$\mu$	$\tau$
leptons in left loop	$\left\{ \begin{array}{l} e \\ \mu \\ \tau \end{array} \right.$	-0.0030194	-0.0016177	-0.0008029
		-0.0016177	-0.0014803	-0.0008029
		-0.0008029	-0.0008029	-0.0006656

Table 2: Numerical evaluation of the non-singlet terms at four-loop order in QED. The last two diagrams of Fig. 3 are typical examples which contribute to this table.

to the truncation of the perturbative series has been completely removed. The analytic

formulae which were presented in this work can be easily used to reevaluate  $\Delta\alpha_{\text{lep}}$ , once more precise values of the input parameters become available.

## 5 Summary and conclusion

We have determined the four-loop QED corrections to the vacuum polarization function in the on-shell scheme in the high energy expansion taking into account the mass hierarchy of all charged leptons in the Standard Model. In order to derive this on-shell result we combine known  $\overline{\text{MS}}$  results for the vacuum polarization function in the massless limit with newly computed contributions at  $q^2 = 0$ , which allow to do the conversion from the  $\overline{\text{MS}}$  to the on-shell scheme. The on-shell result is then used to perform a determination of the leptonic contributions to  $\Delta\alpha$  which describes the running of the effective electromagnetic coupling to the  $Z$ -boson mass scale. The quantity  $\Delta\alpha_{\text{lep}}$  is an important ingredient for constraining the Standard Model in the context of electroweak precision measurements. We perform an numerical evaluation of  $\Delta\alpha_{\text{lep}}$  in order to study the relative size of the different contributions. The size of the new four-loop contribution is small and completely removes the theory uncertainty due to the truncation of the perturbative expansion. The QED corrections to  $\Delta\alpha_{\text{lep}}$  are thus from theory side well under control compared to the error in  $\Delta\alpha_{\text{lep}}$  arising from the uncertainty in the mass of the  $\tau$ -lepton and the  $Z$ -boson mass.

As a by-product of the calculation we also determine the contributions due to heavy fermions to the conversion relation between the  $\overline{\text{MS}}$  and the on-shell scheme for the lepton masses in QED.

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## A The on-shell vacuum polarization function in the high-energy limit at three-loop order and known four-loop results

In this appendix we provide known results for the vacuum polarization function at three-loop [12] and four-loop [23, 24] order from literature in order to have all components in a uniform notation.



## A.1 Three-loop results

The results of Ref. [12] are given here for completeness and read

$$\begin{aligned}
\Pi_A^{(2)}(q^2, M_i) = & \frac{1}{8}L_{qM_i} - \frac{121}{48} - \frac{\pi^2}{3} \left( \frac{5}{2} - 4 \log(2) \right) - \frac{99}{16}\zeta_3 + 10\zeta_5 + \frac{M_i^2}{q^2} \left[ 9L_{qM_i}^2 \right. \\
& - \frac{3}{2}L_{qM_i} + \frac{139}{3} - \pi^2(5 - 8 \log(2)) - \frac{41}{3}\zeta_3 - \frac{70}{3}\zeta_5 \Big] \\
& + \left( \frac{M_i^2}{q^2} \right)^2 \left[ 12L_{qM_i}^3 + \frac{27}{2}L_{qM_i}^2 + L_{qM_i} \left( \frac{115}{4} - 2\pi^2(5 - 8 \log(2)) \right. \right. \\
& - 48\zeta_3) + \frac{437}{6} - \frac{10}{9}\pi^4 - \frac{16}{3}\pi^2 \log^2(2) + \frac{16}{3}\log^4(2) + 128a_4 \\
& \left. \left. + 32\zeta_3 - 20\zeta_5 \right] + \dots, \tag{37}
\end{aligned}$$

$$\begin{aligned}
\Pi_F^{(2)}(q^2, M_i) = & -\frac{1}{6}L_{qM_i}^2 + L_{qM_i} \left( \frac{11}{6} - \frac{4}{3}\zeta_3 \right) - \frac{307}{216} - \frac{4}{9}\pi^2 + \frac{545}{144}\zeta_3 - \frac{M_i^2}{q^2} \left[ 2L_{qM_i}^2 \right. \\
& - \frac{26}{3}L_{qM_i} + \frac{40}{3} + \frac{8}{3}\pi^2 - 16\zeta_3 \Big] - \left( \frac{M_i^2}{q^2} \right)^2 \left[ \frac{4}{3}L_{qM_i}^3 - \frac{19}{3}L_{qM_i}^2 \right. \\
& \left. + L_{qM_i} \left( \frac{8}{9} + \frac{16}{3}\pi^2 - \frac{40}{3}\zeta_3 \right) + \frac{1505}{54} - \frac{352}{9}\zeta_3 \right] + \dots, \tag{38}
\end{aligned}$$

$$\begin{aligned}
\Pi_l^{(2)}(q^2, M_i, M_j) = & \frac{1}{6}L_{qM_i}^2 + \frac{14}{9}L_{qM_i} + L_{qM_j} \left( \frac{5}{18} - \frac{4}{3}\zeta_3 \right) - \frac{1}{3}L_{qM_j}L_{qM_i} - \frac{116}{27} \\
& + \frac{2}{9}\pi^2 + \frac{38}{9}\zeta_3 + 2\frac{M_i^2}{q^2} \left[ L_{qM_i}^2 - 2L_{qM_i}L_{qM_j} + \frac{13}{3}L_{qM_i} - 2 + \frac{2}{3}\pi^2 \right] \\
& - 16\frac{M_j^2}{q^2} \left[ \frac{4}{3} - \zeta_3 \right] - \left( \frac{M_i^2}{q^2} \right)^2 \left[ \frac{83}{54} + \frac{224}{9}\zeta_3 - L_{qM_i} \left( \frac{68}{9} + \frac{8}{3}\pi^2 \right) \right. \\
& - L_{qM_i}^2 \frac{32}{3} - L_{qM_i}^3 \frac{8}{3} - L_{qM_j} \left( \frac{2}{9} + \frac{16}{3}\zeta_3 - \frac{10}{3}L_{qM_i} - 4L_{qM_i}^2 \right) \Big] \\
& + 12\frac{M_i^2}{q^2} \frac{M_j^2}{q^2} [3 - 2L_{qM_i}] - \left( \frac{M_j^2}{q^2} \right)^2 \left[ \frac{50}{9} - 16\zeta_3 - L_{qM_j}^2 \right. \\
& \left. - L_{qM_j} (8\zeta_3 - 4L_{qM_i}) + \frac{26}{3}L_{qM_i} - 2L_{qM_i}^2 \right] + \dots, \tag{39}
\end{aligned}$$

$$\begin{aligned}
\Pi_h^{(2)}(q^2, M_i, M_j) = & -\frac{1}{6}L_{qM_j}^2 + L_{qM_j} \left( \frac{11}{6} - \frac{4}{3}\zeta_3 \right) - \frac{37}{6} + \frac{38}{9}\zeta_3 - 16\frac{M_j^2}{q^2} \left[ \frac{4}{3} - \zeta_3 \right] \\
& - \frac{M_i^2}{q^2} \left[ 2L_{qM_j}^2 - \frac{26}{3}L_{qM_j} + \frac{187}{9} \right] - \left( \frac{M_j^2}{q^2} \right)^2 \left[ \frac{67}{3} - 16\zeta_3 \right. \\
& \left. + L_{qM_j} \left( \frac{26}{3} - 8\zeta_3 \right) + L_{qM_j}^2 \right] + 12\frac{M_i^2}{q^2} \frac{M_j^2}{q^2} [1 - 2L_{qM_j}]
\end{aligned}$$

$$\begin{aligned}
& + \left( \frac{M_i^2}{q^2} \right)^2 \left[ \frac{847}{18} - \frac{224}{9} \zeta_3 - L_{qM_i} \left( \frac{10}{9} - \frac{2}{3} L_{qM_j} \right) \right. \\
& \left. - L_{qM_j} \left( \frac{74}{3} - \frac{16}{3} \zeta_3 \right) + \frac{20}{3} L_{qM_j}^2 - \frac{4}{3} L_{qM_j}^3 \right] + \dots
\end{aligned} \tag{40}$$

For the contributions  $\Pi_A^{(2)}(q^2, M_i)$  and  $\Pi_F^{(2)}(q^2, M_i)$  in Eqs. (37) and (38) we have taken the sub-leading terms in the small mass expansion from Refs. [16, 17, 19, 21], which we have converted in addition from the  $\overline{\text{MS}}$  to the on-shell scheme. Using **q2e** and **exp** [36–38] we have determined also the sub-leading terms for  $\Pi_l^{(2)}(q^2, M_i, M_j)$  and  $\Pi_h^{(2)}(q^2, M_i, M_j)$  in Eqs. (39) and (40). The arising integrals have been computed in two ways: on the one hand they were determined by applying Laporta’s algorithm [39, 40] in order to reduce all integrals to a set of known master integrals, and on the other hand they were calculated by using the **MATAD** and **MINCER** [78–80] routines. The dots in the equations stand for terms of higher order in the small mass expansion. The mass  $M_i$  in Eqs. (39) and (40) is the mass of the lepton in the outer loop which connects to the external photons, whereas the mass  $M_j$  is the mass of the internal lepton loop. The definition of the remaining symbols which appear in Eqs. (37)–(40) can be found in Section 4.

## A.2 Four-loop results

The results for the vacuum polarization function of Refs. [23, 24] are given here for completeness and read

$$\begin{aligned}
\Pi_A^{(3)}(q^2, M_i) &= \frac{23}{32} L_{qM_i} - \frac{71189}{8640} - \frac{157}{18} \pi^2 - \frac{59801}{32400} \pi^4 \\
&+ \pi^2 \log(2) \left( \frac{59}{3} + \frac{424}{675} \pi^2 \right) - \frac{1559}{270} \pi^2 \log^2(2) + \frac{128}{135} \pi^2 \log^3(2) \\
&+ \frac{1559}{270} \log^4(2) - \frac{128}{225} \log^5(2) + \left( \frac{6559}{80} - \frac{\pi^2}{6} \right) \zeta_3 \\
&- \frac{1603}{30} \zeta_5 - 35 \zeta_7 + \frac{6236}{45} a_4 + \frac{1024}{15} a_5 + \mathcal{O}\left(\frac{M_i^2}{q^2}\right),
\end{aligned} \tag{41}$$

$$\begin{aligned}
\Pi_F^{(3)}(q^2, M_i) &= -\frac{1}{12} L_{qM_i}^2 - L_{qM_i} \left( \frac{1}{3} + \frac{19}{3} \zeta_3 - \frac{20}{3} \zeta_5 \right) + \frac{3361}{225} - \frac{179}{81} \pi^2 \\
&- \frac{2161}{2700} \pi^4 + \frac{64}{27} \pi^2 \log(2) - \frac{53}{15} \pi^2 \log^2(2) + \frac{53}{15} \log^4(2) \\
&+ \frac{29129}{450} \zeta_3 - 4 \zeta_3^2 - \frac{250}{9} \zeta_5 + \frac{424}{5} a_4 + \mathcal{O}\left(\frac{M_i^2}{q^2}\right),
\end{aligned} \tag{42}$$

$$\begin{aligned}
\Pi_{FF}^{(3)}(q^2, M_i) &= -\frac{1}{27} L_{qM_i}^3 + L_{qM_i}^2 \left( \frac{11}{18} - \frac{4}{9} \zeta_3 \right) - L_{qM_i} \left( \frac{302}{81} - \frac{76}{27} \zeta_3 \right) \\
&+ \frac{75259}{17010} + \frac{32}{405} \pi^2 - \frac{15109}{5670} \zeta_3 - \frac{20}{9} \zeta_5 + \mathcal{O}\left(\frac{M_i^2}{q^2}\right),
\end{aligned} \tag{43}$$

$$\Pi_{s,d}^{(3)}(q^2, M_i) = -L_{qM_i} \left( \frac{11}{9} - \frac{8}{3} \zeta_3 \right) + \frac{1963}{945} - \frac{2237}{4320} \pi^4 - \frac{73}{36} \pi^2 \log^2(2)$$

$$+ \frac{73}{36} \log^4(2) + \frac{5309}{280} \zeta_3 - \frac{8}{3} \zeta_3^2 + \frac{20}{3} \zeta_5 + \frac{146}{3} a_4 + \mathcal{O}\left(\frac{M_i^2}{q^2}\right). \quad (44)$$

The definition of the different symbols can be found again in Section 4.

## B Decoupling functions

The decoupling functions in Eqs. (17) and (18) are known in QCD to high order in perturbation theory [62–69] and can be used for the QED calculation considered here after inserting the proper values for the color factors. We give the results of the perturbative expansion up to the order needed within this work

$$\begin{aligned} \zeta_\gamma^2(\mu, \bar{\alpha}^{(n_f)}(\mu), \overline{m}_h) &= 1 - \left(\frac{\bar{\alpha}^{(n_f)}(\mu)}{\pi}\right) \frac{n_h}{3} L_{\mu\overline{m}_h} - \left(\frac{\bar{\alpha}^{(n_f)}(\mu)}{\pi}\right)^2 n_h \left(\frac{13}{48}\right. \\ &\quad \left. - \frac{1}{4} L_{\mu\overline{m}_h} - \frac{n_h}{9} L_{\mu\overline{m}_h}^2\right) + \mathcal{O}\left((\bar{\alpha}^{(n_f)})^3\right), \end{aligned} \quad (45)$$

$$\begin{aligned} \zeta_m(\mu, \bar{\alpha}^{(n_f)}(\mu), \overline{m}_h) &= 1 + \left(\frac{\bar{\alpha}^{(n_f)}(\mu)}{\pi}\right)^2 \frac{n_h}{8} \left(\frac{89}{36} - \frac{5}{3} L_{\mu\overline{m}_h} + L_{\mu\overline{m}_h}^2\right) \\ &\quad - \left(\frac{\bar{\alpha}^{(n_f)}(\mu)}{\pi}\right)^3 n_h \left[\frac{683}{576} - \frac{57}{32} \zeta_3 + \frac{11}{360} \pi^4 + \frac{\pi^2}{6} \log^2(2)\right. \\ &\quad \left. - \frac{1}{6} \log^4(2) - 4a_4 + L_{\mu\overline{m}_h} \left(\frac{13}{64} - \frac{3}{4} \zeta_3\right) + \frac{1}{4} L_{\mu\overline{m}_h}^2\right. \\ &\quad \left. + \frac{n_h}{18} \left(\frac{1685}{432} - 7\zeta_3 + \frac{31}{6} L_{\mu\overline{m}_h} - \frac{5}{4} L_{\mu\overline{m}_h}^2 + L_{\mu\overline{m}_h}^3\right)\right. \\ &\quad \left. - \frac{n_l}{9} \left(\frac{1327}{432} - 2\zeta_3 - \frac{53}{16} L_{\mu\overline{m}_h} - \frac{1}{4} L_{\mu\overline{m}_h}^3\right)\right] + \mathcal{O}\left((\bar{\alpha}^{(n_f)})^4\right), \end{aligned} \quad (46)$$

with  $L_{\mu\overline{m}_h} = \log\left(\frac{\mu^2}{\overline{m}_h^2}\right)$ , where  $\overline{m}_h$  is the heavy particle mass renormalized in the  $\overline{\text{MS}}$  scheme which has been integrated out in the effective field theory. The number of active fermions is  $n_f = n_l + n_h$  with  $n_h = 1$  and  $n_l$  is the number of light fermions, which are considered as massless. The definition of the remaining symbols in Eqs. (45) and (46) can be found in Section 3 and 4.

## C The relation between the $\overline{\text{MS}}$ and on-shell scheme for lepton masses

The known relations up to three-loop order in QED for the conversion of the lepton masses from the  $\overline{\text{MS}}$  to the on-shell scheme [54–60] are given here for completeness. The expansion coefficient in Eq. (12) at one-loop order reads

$$c_\ell^{(1)}(M_\ell) = -1 - \frac{3}{4} L_{\mu M_\ell}. \quad (47)$$

The coefficients in Eqs. (13), (15) and (19) at two-loop order are given by

$$c_A^{(2)}(M_\ell) = \frac{7}{128} - \frac{\pi^2}{2} \left( \frac{5}{8} - \log(2) \right) - \frac{3}{4}\zeta_3 + \frac{3}{32}L_{\mu M_\ell} (7 + 3L_{\mu M_\ell}) , \quad (48)$$

$$c_l^{(2)}(M_\ell) = \frac{71}{96} + \frac{\pi^2}{12} + \frac{13}{24}L_{\mu M_\ell} + \frac{1}{8}L_{\mu M_\ell}^2 , \quad (49)$$

$$c_F^{(2)}(M_\ell) = \frac{143}{96} - \frac{\pi^2}{6} + \frac{13}{24}L_{\mu M_\ell} + \frac{1}{8}L_{\mu M_\ell}^2 , \quad (50)$$

$$c_h^{(2)}(M_\ell, M_x) = -\frac{89}{288} + \frac{13}{24}L_{\mu M_x} - \frac{1}{8}L_{\mu M_x}^2 + \frac{1}{4}L_{\mu M_\ell}L_{\mu M_x} , \quad (51)$$

and at three-loop order the coefficients of Eqs. (14), (16) and (20) read

$$\begin{aligned} c_A^{(3)}(M_\ell) = & -\frac{2969}{768} - \frac{613}{192}\pi^2 - \frac{\pi^4}{48} + \frac{29}{4}\pi^2 \log(2) + \frac{1}{2}\log^2(2)\pi^2 \\ & - \frac{1}{2}\log^4(2) - \frac{81}{16}\zeta_3 - \frac{\pi^2}{16}\zeta_3 + \frac{5}{8}\zeta_5 - 12a_4 \\ & - L_{\mu M_\ell} \left( \frac{489}{512} - \frac{15}{64}\pi^2 + \frac{3}{8}\log(2)\pi^2 - \frac{9}{16}\zeta_3 \right) \\ & - \frac{27}{128}L_{\mu M_\ell}^2 - \frac{9}{128}L_{\mu M_\ell}^3 , \end{aligned} \quad (52)$$

$$\begin{aligned} c_l^{(3)}(M_\ell) = & \frac{1283}{576} + \frac{13}{18}\pi^2 - \frac{119}{2160}\pi^4 - \frac{11}{9}\log(2)\pi^2 + \frac{2}{9}\log^2(2)\pi^2 \\ & + \frac{1}{9}\log^4(2) + \frac{55}{24}\zeta_3 + \frac{8}{3}a_4 + L_{\mu M_\ell} \left( \frac{65}{384} + \frac{7}{48}\pi^2 \right. \\ & \left. - \frac{\pi^2}{3}\log(2) - \frac{1}{4}\zeta_3 \right) - \frac{13}{32}L_{\mu M_\ell}^2 - \frac{3}{32}L_{\mu M_\ell}^3 , \end{aligned} \quad (53)$$

$$\begin{aligned} c_F^{(3)}(M_\ell) = & \frac{1067}{576} - \frac{85}{108}\pi^2 + \frac{91}{2160}\pi^4 + \frac{8}{9}\log(2)\pi^2 - \frac{\pi^2}{9}\log^2(2) \\ & + \frac{1}{9}\log^4(2) - \frac{53}{24}\zeta_3 + \frac{8}{3}a_4 - \frac{13}{32}L_{\mu M_\ell}^2 - \frac{3}{32}L_{\mu M_\ell}^3 \\ & - L_{\mu M_\ell} \left( \frac{151}{384} - \frac{\pi^2}{3}(1 - \log(2)) + \frac{1}{4}\zeta_3 \right) , \end{aligned} \quad (54)$$

$$\begin{aligned} c_{Fl}^{(3)}(M_\ell) = & -\frac{5917}{3888} + \frac{13}{108}\pi^2 + \frac{2}{9}\zeta_3 - \frac{1}{18}L_{\mu M_\ell} \left[ \frac{143}{6} - \pi^2 \right. \\ & \left. + L_{\mu M_\ell} \left( \frac{13}{2} + L_{\mu M_\ell} \right) \right] , \end{aligned} \quad (55)$$

$$\begin{aligned} c_{ll}^{(3)}(M_\ell) = & -\frac{2353}{7776} - \frac{13}{108}\pi^2 - \frac{7}{18}\zeta_3 - L_{\mu M_\ell} \left[ \frac{89}{216} + \frac{\pi^2}{18} \right. \\ & \left. + \frac{1}{36}L_{\mu M_\ell} \left( \frac{13}{2} + L_{\mu M_\ell} \right) \right] , \end{aligned} \quad (56)$$

$$c_{FF}^{(3)}(M_\ell) = -\frac{9481}{7776} + \frac{4}{135}\pi^2 + \frac{11}{18}\zeta_3 - L_{\mu M_\ell} \left[ \frac{197}{216} - \frac{1}{9}\pi^2 \right]$$

$$+\frac{1}{36}L_{\mu M_\ell}\left(\frac{13}{2}+L_{\mu M_\ell}\right)\Big], \quad (57)$$

$$\begin{aligned} c_{lh}^{(3)}(M_\ell, M_x) = & -\frac{1327}{3888} + \frac{2}{9}\zeta_3 - L_{\mu M_x}\left[\frac{1}{8} + \frac{\pi^2}{18}\right. \\ & \left. + \frac{1}{12}L_{\mu M_\ell}\left(\frac{13}{3} + L_{\mu M_\ell}\right) - \frac{1}{36}L_{\mu M_x}^2\right], \end{aligned} \quad (58)$$

with  $L_{\mu M} = \log\left(\frac{\mu^2}{M^2}\right)$  and  $M_x > M_\ell$ . The definition of the other symbols can be found in Section 4. The remaining three-loop coefficients which are needed for Eqs. (16) and (20) are given in Eqs. (22) to (25) of Section 4. The complete mass dependence of the contributions from diagrams with massive lepton loop insertions at two- and three-loop order has been determined in Refs. [55, 60]. We restrict ourselves in this section to the contributions needed for this work.

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